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On a problem of Sakai in unbounded derivations

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As a quantization of spaces, especially n-dimensional real lines, Sakai [7] posed the following interesting problem: are there simple C^*-algebras $\mathfrak{A}$ and n^{th} family $\{\delta_j\}_{j=1}^n$ of non approximately bounded pregenerators of $\mathfrak{A}$ such that given a $*$-derivation δ of $\mathfrak{A}$ with $\mathcal{D}(\delta) = \bigcap_{j=1}^n \mathcal{D}(\delta_j)$, there exist $k_1, k_2 \in \mathbb{R}$ and an approximately bounded $*$-derivation δ_0 of $\mathfrak{A}$ with the property that $\delta = \sum_{j=1}^n k_j \delta_j + \delta_0$.

In this note, we show that there is at least one model for two dimensional case. It is nothing but the irrational rotation algebra, namely the C^*-crossed product $C(T) \times_\theta \mathbb{Z}$ of the C^*-algebra $C(T)$ of all continuous functions on the one dimensional torus T by an irrational angle θ. More precisely we have the following:

Theorem 1. Let $\mathfrak{A}_\theta$ be the irrational rotation algebra. Then there exist two non approximately bounded pregenerators δ_1, δ_2 of $\mathfrak{A}_\theta$ such that any

*-derivation δ of $\mathfrak{A}_\theta$ with $\mathcal{D}(\delta) = \mathcal{D}(\delta_1) \cap \mathcal{D}(\delta_2)$ can be expressed as $\delta = k_1\delta_1 + k_2\delta_2 + \delta_0$ for some $k_1, k_2 \in \mathbb{R}$ and an approximately bounded *-derivation δ_0 of $\mathfrak{A}_\theta$.

Remark 1. Suppose $\mathcal{D}(\delta) = \mathcal{D}(\overline{\delta_j})$ ($j = 1$ or 2), then one can show that $\delta = k\overline{\delta_j} + \delta_0$ for some $k \in \mathbb{R}$.

We now state our main theorem as follows:

Theorem 2. Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system where $\mathfrak{A}$ is unital abelian, G is discrete abelian, and α is effective. Suppose $\beta_t = \exp t\delta_0$ $(t \in T)$ commuting with α, and there exists an eigenunitary u for β which generates $\mathfrak{A}$. Then for any *-derivation δ of $\mathfrak{A} \times_\alpha G$ with $\mathcal{D}(\delta) = \mathcal{D}(\tilde{\delta_0}) = \mathcal{D}(\delta_0) \odot_\theta G$ there exist a $k \in \mathbb{R}$, ~~pre-~~ generator δ_1 and an approximately bounded *-derivation δ_2 of $\mathfrak{A} \times_\alpha G$ such that (i) $\mathcal{D}(\delta_j) = \mathcal{D}(\delta)$ $(j=1,2)$ $\delta_1|_{\mathfrak{A}} = 0$, δ_1 commutes with $\tilde{\delta_0}$ (ii) $\delta = k\tilde{\delta_0} + \delta_1 + \delta_2$, where $\mathcal{D}(\delta_0) \odot_\theta G$ is the set of all $\mathcal{D}(\delta_0)$-valued function of G with finite support, and $\tilde{\delta_0}(x)(g) = \delta_0[x(g)]$ $(x \in \mathcal{D}(\delta_0) \odot_\theta G)$.

Remark 2. If $G = \mathbb{Z}$, $\delta_1 = l\delta_1'$ for some $l \in \mathbb{R}$ where δ_1' is independent of δ.

Let $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{A}, H, \beta)$ be two C^*-dynamical systems where α, β commute. Then there is a C^*-dynamical system $(\mathfrak{A} \times_\alpha G, H, \tilde{\beta})$ such that $\tilde{\beta}_h(x)(g) = \beta_h[x(g)]$ $(x \in L^1(G; \mathfrak{A}))$. Then we have the following proposition of fixed point type:

<u>Proposition 3.</u> $(\mathfrak{A} \times_\alpha G)^{\tilde{\beta}} = \mathfrak{A}^\beta \times_\alpha G$

Proof. By definition, $\mathfrak{A}^\beta \times_\alpha G \subset (\mathfrak{A} \times_\alpha G)^{\tilde{\beta}}$. Suppose the inclusion is proper, then $(\mathfrak{A}^\beta \times_\alpha G) \times_{\hat{\alpha}} \hat{G} \subsetneqq (\mathfrak{A} \times_\alpha G)^{\tilde{\beta}} \times_{\hat{\alpha}} \hat{G}$ since $\tilde{\beta}$ commutes with $\hat{\alpha}$. Since $(\mathfrak{A} \times_\alpha G)^{\tilde{\beta}} \times_{\hat{\alpha}} \hat{G} \subset ((\mathfrak{A} \times_\alpha G) \times_{\hat{\alpha}} \hat{G})^{\tilde{\tilde{\beta}}}$, it follows from duality [6,8] that $\mathfrak{A}^\beta \hat{\otimes} \mathcal{C}(L^2(G)) \subsetneqq (\mathfrak{A} \hat{\otimes} \mathcal{C}(L^2(G)))^{\beta \otimes 1}$ which is a contradiction. Q.E.D.

<u>Comment 1.</u> We only consider locally compact abelian groups throughout this note.

In what follows, let δ be a $*$-derivation of $\mathfrak{A} \times_\alpha G$ such that $\mathcal{D}(\delta) = \mathcal{D}(\tilde{\delta}_0)$ where δ_0 is a generator of $\mathfrak{A}$ commuting with α. Suppose δ commutes with $\hat{\alpha}$, and G is discrete. Then $\delta(a) \in \mathfrak{A}$ for $a \in \mathcal{D}(\delta_0)$. Let $(x_n)_n \subset \mathcal{D}(\delta)$ with $x_n \to 0$, $\delta(x_n) \to y \in \mathfrak{A} \times_\alpha G$. Since $x_n = \sum_k a_k^{(n)} \lambda(k)$ $(a_k^{(n)} \in \mathcal{D}(\delta_0))$, using the conditional expectation ε of $\mathfrak{A} \times_\alpha G$ onto $\mathfrak{A}$

one has $\varepsilon(x_n \lambda(g)^*) \to 0$ and $\varepsilon[(\delta(x_n) - y)\lambda(g)^*] \to 0$ for each g in G. Thus $a_g^{(n)} \to 0$ and $\varepsilon[\Sigma_k(\delta(a_k^{(n)})\lambda(k-g) + a_k^{(n)}\delta(\lambda(k))\lambda(g)^* - y_k\lambda(k-g))] \to 0$ where $y = \Sigma_k y_k \lambda(k)$ be the Fourier expansion of y in $\mathfrak{A} \times_\alpha G$ ($y_k \in \mathfrak{A}$). Then $a_g^{(n)} \to 0$ and $\delta(a_g^{(n)}) \to y_g$ for all g in G. Since $\mathcal{D}(\delta|_{\mathfrak{A}}) = \mathcal{D}(\delta_0)$, it follows from Batty's result [2] that $\delta|_{\mathfrak{A}}$ is closable. So $y_g = 0$ for all $g \in G$. Consequently we have the following:

<u>Lemma 4.</u> If G is discrete, any $*$-derivation δ of $\mathfrak{A} \times_\alpha G$ such that (i) $\mathcal{D}(\delta) = \mathcal{D}(\tilde{\delta}_0)$ and (ii) δ commutes with $\hat{\alpha}$ is closable.

<u>Remark 3.</u> In the above lemma, the conclusion is unclear unless the condition (ii) is added.

Now let δ be a $*$-derivation of $\mathfrak{A} \times_\alpha G$ with $\mathcal{D}(\delta) = \mathcal{D}(\tilde{\delta}_0)$. Define $\mathcal{J} = \{ x \in \mathcal{D}(\delta_0) \mid a \mapsto \delta(ax)$ is continuous from $\mathcal{D}(\delta_0)$ into $\mathfrak{A} \times_\alpha G \}$. Since $\delta(a\lambda(g)b) = \delta(\lambda(g))\alpha_g^{-1}(a)b + \lambda(g)\delta(\alpha_g^{-1}(a)b)$ and δ_0 commutes with α, we have $x\lambda(g)b = 0$ for all $g \in G$ and $b \in \mathcal{J}$ if $a_n \in \mathcal{D}(\delta_0) \to 0$ and $\delta(a_n) \to x \in \mathfrak{A} \times_\alpha G$. Then $\varepsilon(x\lambda(g))b = 0$ where ε is the projection of norm one from $\mathfrak{A} \times_\alpha G$ onto $\mathfrak{A}$. So $\varepsilon(x\lambda(g)) \in L(\mathcal{J})$, the left annihilator of $\mathcal{I}$. Since $\mathcal{I}$ is a two sided ideal of $\mathcal{D}(\delta_0)$, it follows from the same way as Longo [4] that $L(\mathcal{I}) = 0$. Thus $\varepsilon(x\lambda(g)) = 0$ for all $g \in G$. Let $x = \sum_g x_g \lambda(g)$ be the Fourier expansion of x. Then $x_g = 0$. So $x = 0$. Then $\delta|_{\mathfrak{A}}$ is closable from $(\mathcal{D}(\delta_0), \|\cdot\|_{\delta_0})$ into $\mathfrak{A} \times_\alpha G$. Therefore we have the following:

Lemma 5. Let δ be a $*$-derivation of $\mathfrak{A} \times_\alpha G$ with $\mathcal{D}(\delta) = \mathcal{D}(\tilde{\delta}_0)$. Then δ is relatively bounded on $\mathcal{D}(\delta_0)$ with respect to δ_0, namely $\|\delta(a)\| \leq K(\|a\| + \|\delta_0(a)\|)$ for all $a \in \mathcal{D}(\delta_0)$, with some positive constant K.

Remark 4. Since δ_0 is a pregenerator, one can not directly apply Longo's result. However the crucial part of the above proof is due to his idea [4].

By the above lemma, let $\beta_t = \exp t\delta_0$ $(t \in \mathbb{R})$. Then there exist derivations $\tilde{\delta}_f$ $(f \in L^1(\mathbb{R}))$ of $\mathfrak{A} \times_\alpha G$ such that (i) $\mathcal{D}(\tilde{\delta}_f) = \mathcal{D}(\tilde{\delta}_0)$ and (ii) $\tilde{\delta}_f = \int_{\mathbb{R}} f(t)\, \tilde{\beta}_t \circ \delta \circ \tilde{\beta}_{-t}\, dt$. In fact, since $\|\delta(a)\| \leq M(\|a\| + \|\delta_0(a)\|)$ for $a \in \mathcal{D}(\delta_0)$,

$$\|\delta\circ\beta_t(a) - \delta\circ\beta_s(a)\| \leq M\{\|\beta_t(a) - \beta_s(a)\| + \|\beta_t\circ\delta_0(a) - \beta_s\circ\delta_0(a)\|\}.$$

So $t \mapsto \delta\circ\beta_t(a)$ is continuous for each $a \in \mathcal{D}(\delta_0)$. Thus $t \mapsto \delta\circ\beta_t(x)$ is also continuous for $x \in \mathcal{D}(\tilde{\delta}_0)$ which gives derivations $\tilde{\delta}_f$ for $f \in L^1(\mathbb{R})$ of $\mathfrak{A}\times_\alpha G$ satisfying (i) and (ii). Similarly, for each $g \in G$ one has a derivation $\hat{\delta}_g$ of $\mathfrak{A}\times_\alpha G$ such that (i) $\mathcal{D}(\hat{\delta}_g) = \mathcal{D}(\tilde{\delta}_0)$ and (ii) $\hat{\delta}_g = \int_{\hat{G}} \langle g, p\rangle\, \hat{\alpha}_p \circ \delta \circ \hat{\alpha}_{-p}\, dp$. Moreover suppose $\beta_t = \exp t\delta_0$ is periodic, then we have that $(\hat{\delta}_1)^{\sim}_0 = (\tilde{\delta}_0)^{\wedge}_1$ commutes with $\hat{\alpha}$, $\tilde{\beta}$. In what follows we treat $*$-derivations of $\mathfrak{A}\times_\alpha G$ with the same domain as $\mathcal{D}(\tilde{\delta}_0)$ commuting with $\hat{\alpha}$ and $\tilde{\beta}$, which are denoted by δ. Since it commutes with $\hat{\alpha}$, it follows from Lemma 4 that it is closable. Hence one may assume that it is closed. Let $x \in C^*(G)$, and $(x_i) \subset \mathcal{D}(\delta)$ which converge to x. Put $y_i = \int_T \tilde{\beta}_t(x_i)\, dt \in \mathfrak{A}\times_\alpha G$. Since δ commutes with $\tilde{\beta}$ and δ is closed, $y_i \in \mathcal{D}(\delta) \cap (\mathfrak{A}\times_\alpha G)^{\tilde{\beta}}$ and $y_i \to x$ since $(\mathfrak{A}\times_\alpha G)^{\tilde{\beta}} = C^*(G)$ by Proposition 1. So $\delta|_{C^*(G)}$ is a closed $*$-derivation of $C^*(G)$ since $\delta(y_i) \in C^*(G)$. Since $\hat{\alpha}_p \circ \delta \circ \hat{\alpha}_{-p} = \delta$ for $p \in \hat{G}$ and $\mathcal{F}\circ\hat{\alpha}_p\circ\mathcal{F}^{-1} = \tau_p$ on $C(\hat{G})$, $\hat{\delta} = \mathcal{F}\circ\delta\circ\mathcal{F}^{-1}$ commutes with τ on $C(\hat{G})$ where $\mathcal{F}$ is the Fourier isomorphism of $C^*(G)$ onto $C(\hat{G})$, and τ is the shift action of $\hat{G}$ on $C(\hat{G})$. It follows from Goodman-Nakazato [3, 5] that there exists a one parameter subgroup (p_t) of $\hat{G}$ such that $\hat{\delta}(f)(p) = \lim_{t\to 0} t^{-1}(f(p_t p)$

$- f(P))$ for all $f \in \mathcal{D}(\hat{\delta})$. Since $<g, \cdot> \in \mathcal{D}(\hat{\delta})$, one has $\delta(\lambda(g)) = \partial(g)\lambda(g)$ for all $g \in G$ where $\partial(g) = \lim_{t \to 0}(<g, P_t> - 1)$. Let $\delta_1(a\lambda(g)) = \partial(g)a\lambda(g)$ for all $a \in \mathcal{D}(\delta_0)$ and $g \in G$. Then it is a pregenerator of $\mathfrak{A} \times_\alpha G$ such that $\mathcal{D}(\delta_1) = \mathcal{D}(\tilde{\delta}_0)$ and $\delta_1/\mathfrak{A} = 0$, δ_1 commutes with $\tilde{\delta}_0$. Since δ is a closed *-derivation of $\mathfrak{A} \times_\alpha G$ and $\delta/\mathfrak{A}$ commutes with $\beta_t = \exp t\delta_0$, it follows from Batty [1] that $\delta/\mathfrak{A} = k\delta_0$ for some $k \in \mathbb{R}$. Therefore we have that $\delta(a\lambda(g)) = k\delta_0(a)\lambda(g) + a\partial(g)\cdot\lambda(g) = (k\tilde{\delta}_0 + \delta_1)(a\lambda(g))$, which implies the following lemma:

<u>Lemma 6.</u> Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system where $\mathfrak{A}$ is unital abelian and G is discrete abelian. Let $\beta_t = \exp t\delta_0$ be a periodic action of $\mathbb{R}$ on $\mathfrak{A}$. Suppose β is ergodic, then given a * derivation δ of $\mathfrak{A} \times_\alpha G$ with the property that (i) $\mathcal{D}(\delta) = \mathcal{D}(\tilde{\delta}_0)$ and (ii) δ commutes with $\hat{\alpha}, \tilde{\beta}$, there exist a $k \in \mathbb{R}$ and a pregenerator δ_1 of $\mathfrak{A} \times_\alpha G$ such that (i) $\mathcal{D}(\delta_1) = \mathcal{D}(\delta)$, $\delta_1/\mathfrak{A} = 0$, δ_1 commutes with $\tilde{\delta}_0$, and (ii) $\delta = k\tilde{\delta}_0 + \delta_1$ on $\mathcal{D}(\tilde{\delta}_0)$.

<u>Remark 5.</u> The pregenerator δ_1 defined above would

be written as $\delta_1 = \gamma\delta_1'$ for some $\gamma \in \mathbb{R}$ where δ_1' is not depending on δ. Actually if $G = \mathbb{Z}$, we have $\delta_1'(a\lambda(n)) = in\,a\lambda(n)$ for $a \in \mathcal{D}(\delta_0)$ and $n \in \mathbb{Z}$.

Let δ be a linear mapping from a $*$-subalgebra $\mathcal{D}(\delta)$ of $\mathfrak{A}$ into $\mathfrak{A}$ such that $\delta(ab) = \delta(a)\alpha_g(b) + a\delta(b)$ for all $a, b \in \mathcal{D}(\delta)$ where $g \neq e \in G$ is a fixed element. Suppose there is a unitary u of $\mathcal{D}(\delta)$ such that $1 \notin sp(\alpha_g(u)u^*)$, then we have by direct computation that $\delta(u^n) = \sum_{k=0}^{n-1} \alpha_g(u^k)u^{-k}\delta(u)u^{n-1}$. Since $1 \notin sp(\alpha_g(u)u^*)$, one has that $\sum_{k=0}^{n-1} \alpha(u^k)u^{-k} = (\alpha_g(u^n)u^{-n} - 1)(\alpha_g(u)u^* - 1)^{-1}$. So $\delta(u^n) = \delta(u)u^*(\alpha_g(u)u^* - 1)^{-1}(\alpha_g - id)(u^n) = \delta(u)(\alpha_g(u) - u)^{-1}(\alpha_g - id)(u^n)$ for all $n \in \mathbb{Z}$ since $\delta(1) = 0$. Put $a_g = \delta(u)(\alpha_g(u) - u)^{-1} \in \mathfrak{A}$. Since $a_g(\alpha_g - id)$ is bounded on $\mathfrak{A}$, the conclusion follows. Namely we have the following:

<u>Lemma 7.</u> Suppose $\mathfrak{A}$ is unital abelian and G is discrete. Let δ be a linear mapping of a $*$-subalgebra $\mathcal{D}(\delta)$ of $\mathfrak{A}$ into $\mathfrak{A}$ such that $\delta(ab) = \delta(a)\alpha_g(b) + a\delta(b)$ for $a, b \in \mathcal{D}(\delta)$ for a fixed $g \neq e$. Suppose there exists a unitary $u \in \mathcal{D}(\delta)$ such that $1 \notin sp(\alpha_g(u)u^*)$, then $\delta = a_g(\alpha_g - id)$ on $\mathcal{D}(\delta) \cap C^*(u)$

for some $a_g \in \mathfrak{A}$.

Remark 6. By the above lemma, there is no unbounded α_g- cocycle closed *-derivation if α_g has an eigen unitary generating $\mathfrak{A}$.

Now let $\hat{\delta}_g$ $(g \in G)$ be a derivation of $\mathfrak{A} \times_\alpha G$ as in the previous way (following to Remark 4). Then it implies that $\delta = \Sigma_g \hat{\delta}_g$ on $\mathcal{D}(\delta)$. In fact, let $\delta(a) = \Sigma_g \delta(a)(g)\lambda(g)$ and $\delta(\lambda(h)) = \Sigma_g \delta(\lambda(h))(g)\lambda(g)$ be the Fourier expansion of $\delta(a)$ and $\delta(\lambda(h))$ respectively. Then $\hat{\delta}_g(a) = \delta(a)(g)\lambda(g)$ and $\hat{\delta}_g(\lambda(h)) = \delta(\lambda(h))(g+h)\lambda(g+h)$. ~~Suppose~~ δ commutes with $\tilde{\beta}$, it follows from Lemma 6 that $\hat{\delta}_e = k\tilde{\delta_0} + \delta_1$ on $\mathcal{D}(\delta)$ where k, δ_1 are as in Lemma 6. Let $\delta_g(a) = \hat{\delta}_g(a)\lambda(g)^*$ for $a \in \mathcal{D}(\delta_0)$ $(g \neq e)$. Then δ_g satisfy the condition of Lemma 7. Suppose there exists a unitary $u \in \mathcal{D}(\delta_0)$ such that (i) $1 \notin Sp(\alpha_g(u)u^*)$ $(g \neq e)$ and (ii) $\mathfrak{A} = C^*(u)$. Since δ commutes with $\tilde{\beta}$, and α commutes with $\beta_t = \exp t\delta_0$ which is ergodic, we have $a_g \in \mathbb{C}1$. Then $\hat{\delta}_g(a) = a_g(\alpha_g - id)(a)\lambda(g) = [a_g\lambda(g), a]$. Hence $\hat{\delta}_g(a\lambda(h)) = \hat{\delta}_g(a)\lambda(h) + a\hat{\delta}_g(\lambda(h)) = [a_g\lambda(g), a\lambda(h)] + a\hat{\delta}_g(\lambda(h))$. Since $\hat{\delta}_g - ad(a_g\lambda(h))$ is a derivation on $\mathcal{D}(\delta)$, one has $\hat{\delta}_g(\lambda(h)) = 0$ for $h \in G$.

In fact, since $\hat{\delta}_g(\lambda(h)) = \delta(\lambda(h))(g+h)\,\lambda(g+h)$, we have that $\delta(\lambda(h+k))(h+k+g)\,u = \delta(\lambda(h))(h+g)\,\alpha_g(u) + \delta(\lambda(k))(k+g)\cdot u$ for all $h, k \in G$. Since $1 \in \mathcal{D}(\delta)$, we have $\delta(1)(g)=0$. So $\delta(\lambda(h))(h+g) = 0$ for all $h \in G$ or $\alpha_g(u) = u$. Since $1 \notin sp(\alpha_g(u)u^*)$, we have $\delta(\lambda(h))(h+g) = 0$ for all $h \in G$. Consequently $\delta = k\tilde{\delta}_0 + \delta_1 + \Sigma_{g\neq e}\, ad\,(a_g\lambda(g))$ on $\mathcal{D}(\delta)$. Let $\delta_F = ad\,(\Sigma_{g\in F}\, a_g\lambda(g))$ for a finite set F of $G-\{e\}$ with $F=-F$. Then δ_F are bounded $*$-derivations of $\mathfrak{A}\times_\alpha G$ such that $\delta_F(\lambda(h)) = 0$ and δ_F converges to δ_2 pointwisely on $\mathcal{D}(\delta)$ where $\delta_2(a\lambda(h)) = \Sigma_{g\neq e}[a_g\lambda(g), a\lambda(h)]\ (= (\delta - \hat{\delta}_e)(a)\lambda(h))$. Then $\delta = k\tilde{\delta}_0 + \delta_1 + \delta_2$ on $\mathcal{D}(\delta)$ and $\delta_2(\lambda(g)) = 0$ for all $g \in G$, which implies the following proposition:

<u>Proposition 8.</u> Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system where $\mathfrak{A}$ is unital abelian and G is discrete. Let $\beta_t = \exp t\delta_0$ be an ergodic action of T on $\mathfrak{A}$ commuting with α. Suppose there exists a unitary $u \in \mathcal{D}(\delta_0)$ such that (i) $1 \notin sp(\alpha_g(u)u^*)$ $(g\neq e)$, (ii) $\mathfrak{A} = C^*(u)$, then given a $*$-derivation δ of $\mathfrak{A}\times_\alpha G$ such that (i) $\mathcal{D}(\delta) = \mathcal{D}(\tilde{\delta}_0)$ and (ii) δ commutes with $\tilde{\beta}$, there exist a $k \in \mathbb{R}$, a pregenerator δ_1 and an approximately bounded $*$-derivation δ_2 of $\mathfrak{A}\times_\alpha G$ such that (i) $\mathcal{D}(\delta_1) = \mathcal{D}(\delta)$, $\delta_1|_{\mathfrak{A}} = 0$, δ_1 commutes with $\tilde{\delta}_0$,

ii) $\mathcal{D}(\delta_2) = \mathcal{D}(\delta)$, $\delta_2(\lambda(g)) = 0$ for all $g \in G$, and iii) $\delta = k\tilde{\delta}_0 + \delta_1 + \delta_2$.

Remark 7. In the case of discrete abelian groups, the Fourier expansion of any element of $\mathfrak{A} \times_\alpha G$ can be taken in the uniform sense. In fact, taking a net $\{f_i\}$ of positive definite functions on G with finite support converging to 1, one can show that $\sum_g f_i(g)\, a_g \lambda(g)$ converges to $\sum_g a_g \lambda(g) \in \mathfrak{A} \times_\alpha G$ uniformly.

Proof of Theorem 2 : Since β commutes with α and β is ergodic, we have $\alpha_g(u)u^* \in \mathbb{C}1$. Since $\mathfrak{A} = C^*(u)$ and α is effective, there are $c_g \neq 1$ $(g \neq e)$ such that $\alpha_g(u) = c_g u$. So $1 \notin sp(\alpha_g(u)u^*)$ $(g \neq e)$. Let $\tilde{\delta}_n = \int_T e^{-int} \tilde{\beta}_t \circ \delta \circ \tilde{\beta}_{-t}\, dt$ on $\mathcal{D}(\delta)$ for $n \in \mathbb{Z}$. Since $\tilde{\delta}_0$ commutes with $\tilde{\beta}$, it follows from Proposition 8 that $\tilde{\delta}_0 = k'\tilde{\delta}_0 + \delta_1' + \delta_2'$ where δ_i', k' are as in Proposition 8. Since $\tilde{\beta}_t \circ \tilde{\delta}_n \circ \tilde{\beta}_{-t} = e^{int}\tilde{\delta}_n$ $(n \in \mathbb{Z})$, $\tilde{\beta}_t \circ \tilde{\delta}_n(\lambda(g)) = e^{itn}\tilde{\delta}_n(\lambda(g))$. Since $\beta_t(u^n) = e^{itn}u^n$, we have that $u^{-n}\tilde{\delta}_n(\lambda(g)) \in (\mathfrak{A} \times_\alpha G)^{\tilde{\beta}} = C^*(G)$. So there are $b(n,g) \in C^*(G)$ such that $\tilde{\delta}_n(\lambda(g)) = u^n b(n,g)$. Let $\delta(\lambda(g)) = \sum_h \delta(\lambda(g))(h)\, \lambda(h)$ and $b(n,g) = \sum_h b(n,g)(h)\, \lambda(h)$ be the Fourier expansion of $\delta(\lambda(g))$ and $b(n,g)$ respectively.

Since $\mathfrak{A} = C^*(u)$ and $\beta_t(u) = e^{it}u$, we have that $\delta(\lambda(g))(h) = a(0) + \sum_{n\neq 0} b(n,g)(h)\, u^n$ where $a(0)$ is the 0-component of the expansion of $\delta(\lambda(g))(h)$ in $\mathfrak{A}$. Since $\tilde{\delta}_0 = k'\tilde{\delta}_0 + \delta_1' + \delta_2'$, one has $\tilde{\delta}_0(\lambda(g)) = \partial(g)\lambda(g)$. By unicity, $\int_T \beta_t(\delta(\lambda(g))(h))dt = \partial(g)1$ $(g=h)$, $=0$ (otherwise), which is nothing but $a(0)$. Therefore we deduce that $\delta(\lambda(g)) = \partial(g)\lambda(g) + \sum_h \sum_{n\neq 0} b(n,g)(h) \times u^n \lambda(h) = \partial(g)\lambda(g) + \sum_{n\neq 0} u^n b(n,g) = \partial(g)\lambda(g) + \sum_{n\neq 0} \tilde{\delta}_n(\lambda(g))$. Moreover $\delta(a) = \sum_g \hat{\delta}_g(a)$ for all $a \in \mathcal{D}(\delta_0)$. It follows from Lemma 7 that $\hat{\delta}_g(a) = f_g(\alpha_g - id)(a)\lambda(g)$ for some $f_g \in \mathfrak{A}$ $(g\neq e)$. So $\hat{\delta}_g(a) = [f_g\lambda(g), a]$ for all $a \in \mathcal{D}(\delta_0)$. Since $\hat{\delta}_e$ commutes with $\hat{\alpha}$, we have $\hat{\delta}_e(a) \in \mathfrak{A}$ for all $a \in \mathcal{D}(\delta_0)$. Since $(\hat{\delta}_e)_0^{\sim}$ commutes with $\hat{\alpha}$ and $\tilde{\beta}$, it means that $(\hat{\delta}_e)_0^{\sim} = k\tilde{\delta}_0 + \delta_1$ where k, δ_1 are as in Lemma 6. Then $\int_T e^{-it}\beta_t \circ \hat{\delta}_e(u)\,dt = k\delta_0(u)$. Since $\beta_t(u) = e^{it}u$, we have $\delta_0(u) = iu$. Let $\hat{\delta}_e(u) = \sum_n a_n u^n$ $(a_n \in \mathbb{C})$. Then $a_1 = ik$. Therefore $\hat{\delta}_e(u) = k\delta_0(u) + \sum_{n\neq 1} a_n u^n$. Since $\hat{\delta}_e$ is a $*$-derivation, we deduce that $\hat{\delta}_e(u^n) = n\hat{\delta}_e(u)u^{n-1} = kn\delta_0(u)u^{n-1} + \sum_{m\neq 1} n a_m u^{m+n-1} = k\delta_0(u^n) + \sum_{m\neq 1} n a_m u^{m+n-1}$. Hence $\hat{\delta}_e(u^n)\lambda(g) = k\tilde{\delta}_0(u^n\lambda(g)) + \sum_{m\neq 1} n a_m u^{m+n-1}\lambda(g)$. Consequently, we have that $\delta(u^n\lambda(g)) = \delta(u^n)\lambda(g) + u^n\delta(\lambda(g))$
$= (k\tilde{\delta}_0 + \delta_1)(u^n\lambda(g)) + \sum_{h\neq e}[f_h\lambda(h), u^n]\lambda(g) + \sum_{m\neq 0} u^n \tilde{\delta}_m(\lambda(g))$

$+ \sum_{m\neq 0} n a_{m+1} u^{n+m} \lambda(g)$. Since $\delta - k\tilde{\delta}_0 - \delta_1$ is a $*$-derivation, so is $\sum_{h\neq e} [f_h \lambda(h), u^n] \lambda(g) + \sum_{m\neq 0} u^{n+m} \tilde{S}_m(\lambda(g)) + \sum_{m\neq 0} n a_{m+1} u^{n+m} \lambda(g)$. Since $ad(f_h\lambda(h))(u^n)\lambda(g) + u^n\, ad(f_h\lambda(h))(\lambda(g)) = ad(f_h\lambda(h))(u^n\lambda(g))$, we deduce that $u^n(\sum_{m\neq 0} \tilde{S}_m(\lambda(g)) - \sum_{h\neq e}[f_h\lambda(h), \lambda(g)]) + \sum_{m\neq 0} n a_{m+1} u^{n+m} \times \lambda(g)$ is a $*$-derivation. Let $a = \sum_{m\neq 0} a_{m+1} u^m \in \mathfrak{A}$. Conventionally put $\sigma(\lambda(g)) = \sum_{m\neq 0} \tilde{S}_m(\lambda(g)) - \sum_{h\neq e} [f_h\lambda(h), \lambda(g)]$. Moreover, put $\Delta(u^n\lambda(g)) = u^n\sigma(\lambda(g)) + n a u^n \lambda(g)$. Since $\delta_0(u^n) = inu$, we see $nau^n\lambda(g) = (-i)a\tilde{\delta}_0(u^n\lambda(g))$. Now since $\Delta(u^n\lambda(g)\, u^m\lambda(h))$ $= \Delta(u^n\lambda(g))u^m\lambda(h) + u^n\lambda(g)\Delta(u^m\lambda(h))$, we can show that $u^{n+m}(\sigma(\lambda(g))\lambda(h) - \lambda(g)\sigma(\lambda(h))) = m(\alpha_g(a) - a)\, u^{n+m}\lambda(g+h)$. Put $h=e$ and $m=1$. Then we have $u^n\sigma(\lambda(g)) = (\alpha_g(a)-a)u^n\lambda(g)$ for all $n\in\mathbb{Z}$ and $g\in G$. Therefore $\Delta(u^n\lambda(g)) = (\alpha_g(a)-a)u^n\lambda(g)$ $+ nau^n\lambda(g) = (\alpha_g(a) + (n-1)a)\, u^n\lambda(g)$. Since Δ is a derivation, we get $\alpha_g(a) = a$ for all $g\in G$. So $a = z1$ for some $z \in \mathbb{C}$. Then $\Delta(u^n\lambda(g)) = znu^n\lambda(g) = iz\tilde{\delta}_0(u^n\lambda(g))$. Finally, we obtain that $\delta(u^n\lambda(g)) = (c\tilde{\delta}_0 + \delta_1)(u^n\lambda(g)) + \sum_{h\neq e}[f_h\lambda(h), u^n\lambda(g)]$ for some $c\in\mathbb{R}$. Let $\delta_F(a\lambda(g)) = \sum_{h\in F}[f_h\lambda(h), a\times \lambda(g)]$ for $a\in\mathcal{D}(\delta_0)$ and $g\in G$ where F is a finite set of $G-\{e\}$ with $F=-F$. Then δ_F is a bounded $*$-derivation of $\mathfrak{A}\times_\alpha G$ for all F and $\delta_F \to \delta_2$ pointwisely. Hence δ_2 is approximately bounded. This completes the proof.

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